Student's Name……………………….…….. **Student Number:………..…………….**

PRESBYTERIAN LADIES' COLLEGE SYDNEY 1888

2010
TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 120
- Attempt questions 1-8
- All questions are of equal value

1	2	3	4	5	6	7	8	Total	Total
								/120	**%**

Question 1 (15 marks) Start a new sheet of writing paper.

Marks

a) Find

$$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x \, dx .$$ **2**

b) Find $\int \frac{dx}{\sqrt{x^2 - 6x + 8}} dx$ **2**

c) Use partial fractions to show **3**

$$\int_2^5 \frac{2x+2}{(x-1)(2x-1)} dx = \log_e\left(\frac{256}{27}\right).$$

d) Find $\int \sin(\log_e x) \, dx$. **4**

e) Use the substitution $t = \tan\frac{x}{2}$ to find **4**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(3\cos x + 4\sin x + 5)} .$$

End of Question 1

Question 2 **(15 marks)** **Start a new sheet of writing paper.** **Marks**

a) If $z=\sqrt{3}+i$ and $w=1-i$

i) Write $\dfrac{z}{w}$ in the form $a+ib$ where a and b are real. **1**

ii) Write $\dfrac{z}{w}$ in mod-arg form. **2**

iii) Hence, or otherwise, show that $\cos\dfrac{5\pi}{12}=\dfrac{\sqrt{6}-\sqrt{2}}{4}$ **2**

iv) Express $\left(\dfrac{z}{w}\right)^{12}$ in the form $a+ib$ where a and b are real. **2**

b) The points *Z*, *W* and *O* on the Argand diagram represent the complex numbers z, w and o respectively. If $z=3+i$ and $o=0+0i$. Find the complex number w, in $a+ib$ form where a and b are real, if ΔOZW in anti-clockwise order, is right-angled at *Z* and the distance from *Z* to *W* is twice the distance from *O* to *Z*. **2**

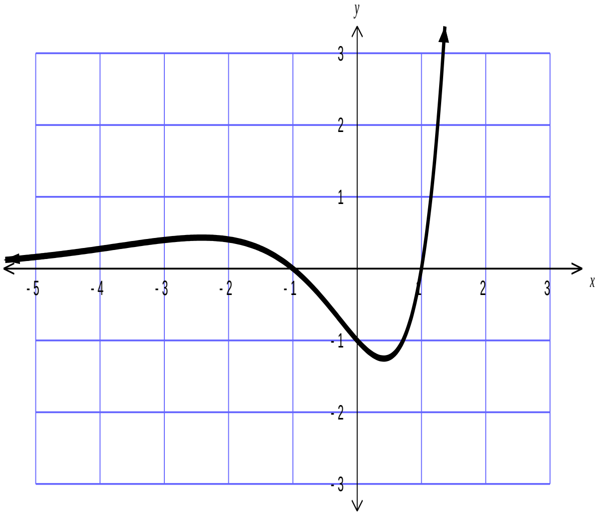
c) The point P on the Argand diagram represents the complex number $z=x+iy$ which satisfies $(z)^2=2-(\overline{z})^2$. Find the equation of the locus of P in terms of *x* and *y*. What type of curve is this locus? **3**

d) If *z* is a complex number such that $z=r(\cos\theta+i\sin\theta)$, where *r* is real, show that $\arg(z+r)=\dfrac{1}{2}\theta$. **3**

End of Question 2

Question 3 (15 marks) **Start a new sheet of writing paper.** **Marks**

a) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

i) $y = \dfrac{1}{f(x)}$ **1**

ii) $|y| = f(x)$ **1**

iii) $y = \left[f(x)\right]^2$ **2**

iv) $y = \sqrt{f(x)}$ **2**

v) $y = x\left(f(x)\right)$ **2**

b) i) Express the complex number $1+i$ in the form $r(\cos\theta + i\sin\theta)$. **1**

ii) Hence prove that $(1+i)^n + (1-i)^n = 2(2^{\frac{n}{2}}\cos\frac{n\pi}{4})$ where n is a positive integer. **3**

iii) If $(1+x)^n = p_0 + p_1x + p_2x^2 + \ldots + p_nx^n$, prove that **3**

$$p_0 - p_2 + p_4 - \ldots.. = 2^{\frac{n}{2}}\cos\frac{n\pi}{4} \text{ and } p_1 - p_3 + p_5 - \ldots. = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}.$$

End of Question 3

Question 4 **(15 marks)** **Start a new sheet of writing paper.**

Marks

a) Consider the hyperbola H with equation $\dfrac{(x-1)^2}{16}-\dfrac{(y+3)^2}{25}=1$

i) Find the centre, the eccentricity and the co-ordinates of the foci of *H*. **3**

ii) Write down the equations of the directrices and the asymptotes of *H*. **3**

iii) Sketch *H* showing all of the above features. **1**

b) Using the focus-directrix definition of the ellipse, centred at the origin, prove that the sum of the focal lengths is constant. **2**

c) Given the hyperbola $x^2-y^2=a^2$

i) Show that $(a\sec\theta,\ a\tan\theta)$ are the parametric coordinates of a point on the hyperbola. **1**

ii) Show that the equation of the tangent to $x^2-y^2=a^2$ at $(a\sec\theta,\ a\tan\theta)$ is $x-y\sin\theta=a\cos\theta.$ **2**

iii) Prove that the area of the triangle bounded by a tangent and the asymptotes is a constant. **3**

End of Question 4

Question 5 **(15 marks)** **Start a new sheet of writing paper.**

Marks

a) Consider $P(x) = x^4 - 6x^2 - 8x - 3$.

i) Given that $P(x)$ has a zero of multiplicity 3, express $P(x)$ as a product of linear factors. **2**

ii) Sketch the graph of $P(x)$. **2**

b) $P(x)$ is a polynomial of the form $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real. **4**
$P(x)$ has roots of 5 and i and when divided by $(x-2)$ the remainder is 15.
Find $P(x)$.

c) i) Show that $(1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n = (1-\sqrt{x})^{n-1}\sqrt{x}.$ **1**

ii) If $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$ for $n \geq 0$ show that $I_n = \dfrac{n}{n+2} I_{n-1}$ for $n \geq 1$. **2**

iii) Hence show that $\dfrac{1}{I_n} = {}^{n+2}C_n$ for $n \geq 0$. **2**

d) If $a > 0,\ b > 0,\ c > 0,\ d > 0,$ show that $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{d} + \dfrac{d}{a} \geq 4$ **2**

End of Question 5

Question 6 **(15 marks)** **Start a new sheet of writing paper.**

Marks

a) i) Prove that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ **3**

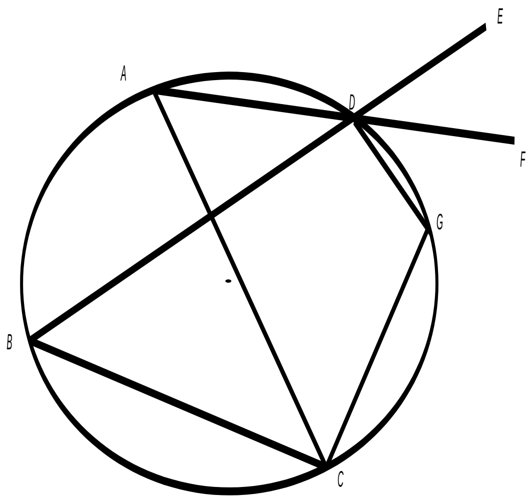
ii) Show that roots of the equation $16x^4 - 20x^2 + 5 = 0$ are $x = \sin\frac{k\pi}{5}$, where $k = 1, 2, 3, 4$. **2**

iii) Construct an equation whose roots are each 1 greater than those of $16x^4 - 20x^2 + 5 = 0$ **2**

iv) Hence or otherwise find the exact value of $\sum_{k=1}^{4} \frac{1}{1 + \sin\frac{k\pi}{5}}$ **2**

b) Find the equation of the tangent to the curve $5x^2 - 6xy + y^2 - 2x + 4y - 3 = 0$ at the point $(1, 2)$. **3**

c)



AC bisects $\angle BCG$ **3**

ADF and BDE are straight lines.

Prove that FD bisects $\angle EDG$.

End of Question 6

Question 7 (15 marks) **Start a new sheet of writing paper.**

Marks

a) The n^{th} derivative of $f(x)$ is $\dfrac{d^n}{dx^n} f(x) = \dfrac{d^{n-1}}{dx^{n-1}}\left[\dfrac{d}{dx} f(x)\right]$.

i) Show that $\dfrac{d^n}{dx^n}(x^n) = n!$ **2**

ii) Prove, by mathematical induction, that for all positive integers, n **3**

$$\frac{d^n}{dx^n}(x^n \ln x) = n!(\ln x + \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n})$$

b) A railway line has been constructed around a circular curve of radius 500m. The distance between the rails is 1.5m and the outside rail is 0.1m above the inside rail.

i) Draw a diagram showing all forces on the train. **1**

ii) Show that $tan\,\theta = \dfrac{v^2}{gr}$, given that there is no sideways force on the wheels for a train on this curve. **2**

iii) Find the optimal speed for the train around this curve. (Take $g = 9.8\ m/s^2$) **1**

c) A particle of mass *m* projected vertically upwards with initial speed *u* m/s experiences a resistance of magnitude *Kmv* Newtons when the speed is *v* m/s where *K* is a positive constant. After T seconds the particle attains its maximum height *h*. Let the acceleration due to gravity be $g\ m/s^2$.

i) Show that the acceleration of the particle is given by $\ddot{x} = -(g + Kv)$ where x is the height of the particle t seconds after the launch. **1**

ii) Prove that T is given by $T = \dfrac{1}{K}\log_e\left(\dfrac{g + Ku}{g}\right)$ seconds. **2**

iii) Prove that h is given by $h = \dfrac{u - gT}{K}$ metres. **3**

End of Question 7

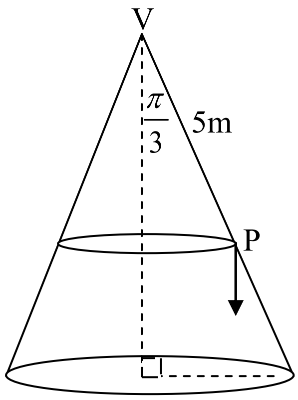
Question 8 **(15 marks)** **Start a new sheet of writing paper.** **Marks**

a) A particle is projected from the origin with initial velocity U to pass through a point (a,b).

i) Show that the Cartesian equation of the motion of the particle is given by $y = \frac{-gx^2}{2U^2}\sec^2\alpha + x\tan\alpha$. You must DERIVE all equations of motion. **3**

ii) Prove that there are two possible trajectories if: **3**
$\left(U^2 - gb\right)^2 > g^2\left(a^2 + b^2\right)$

b) A circular cone of semi-vertical angle $\frac{\pi}{3}$ is fixed with its vertex upwards. A particle P of mass *m* kg is attached to the vertex at *V* by a light inextensible string of length 5m. The particle *P* rotates with uniform angular velocity ω rad/sec in a horizontal circle whose centre is vertically below *V*, on the outside surface of the cone and in contact with it. Let *T* be the tension in the string, *N* the normal reaction force and *mg* the gravitational force at *P*.



i) Resolve the forces on *P* in the horizontal and vertical directions. **3**

ii) Show that $T = \frac{m}{4}\left(2g + 15\omega^2\right)$ and find a similar expression for *N*. **4**

iii) Show that for the particle to remain in uniform circular motion on the surface of the cone, then $\omega^2 < \frac{2g}{5}$, where *g* is the acceleration due to gravity. **2**

End of Examination

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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PLC Sydney Maths Department

Ver 1

Solutions for exams and assessment tasks

Academic Year	Yr 12	Calendar Year	2010
Course	Ext. 2	Name of task/exam	Trial Exam

Question 1

a) $\int \tan^4 x \sec^2 x\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$= \int u^4\,du$

$= \frac{u^5}{5} + c$

$= \frac{\tan^5 x}{5} + c$

b) $\int \frac{dx}{\sqrt{x^2-6x+8}}$

$= \int \frac{dx}{\sqrt{(x-3)^2-1}}$

$= \ln\left(x-3+\sqrt{x^2-6x+8}\right)+c$

c) $\int \frac{2x+2}{(x-1)(2x-1)}\,dx$

$\frac{2x+2}{(x-1)(2x-1)} = \frac{A}{x-1} + \frac{B}{2x-1}$

$= \frac{4}{x-1} - \frac{6}{2x-1}$

$\therefore \int \frac{2x+2}{(x-1)(2x-1)}\,dx = \int \frac{4}{x-1} - \frac{6}{2x-1}\,dx$

$= 4\ln(x-1) - 3\ln(2x-1)$

$= \left[\ln\frac{(x-1)^4}{(2x-1)^3}\right]_2^5$

$= \ln\frac{4^4}{9^3} - \ln\left(\frac{1}{3^3}\right)$

$= \ln \frac{4^4}{3^6} \times \frac{3^3}{1}$

$= \ln \frac{4^4}{3^3}$

$= \ln\left(\frac{256}{27}\right)$

e) $\int_0^{\frac{\pi}{2}} \frac{dx}{3\cos x + 4\sin x + 5}$

$t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}$

$dx = \frac{2}{\sec^2\frac{x}{2}}$

$= \frac{2}{1+t^2}$

$x = 0,\ t = 0$

$x = \frac{\pi}{2}\ \ t = 1$

$$= \int_0^1 \frac{\frac{2}{1+t^2}\,dt}{3\left(\frac{1-t^2}{1+t^2}\right)+4\left(\frac{2t}{1+t^2}\right)+5}$$

$$= \int_0^1 \frac{2}{3-3t^2+8t+5+5t^2}\,dt$$

$$= \int_0^1 \frac{2\,dt}{2t^2+8t+8}$$

$$= \int_0^1 \frac{\cancel{2}\,1}{t^2+4t+4}\,dt$$

$$= \int_0^1 \frac{1}{(t+2)^2}\,dt$$

$$= \left[\frac{-1}{t+2}\right]_0^1$$

$$= \left[\frac{-1}{3}+\frac{1}{2}\right]$$

$$= \frac{1}{6}$$

d) $\int \sin(\log x)\,dx$

$u = \sin(\log x) \qquad v = x$

$du = \frac{1}{x}\cos(\log x) \qquad dv = 1$

$= x\sin(\log x) - \int x \cdot \frac{1}{x}\cos(\log x)\,dx$

$= x\sin(\log x) - \int \cos(\log x)\,dx$

$u = \cos(\log x) \qquad v = x$

$du = \frac{-1}{x}\sin(\log x) \qquad dv = 1$

$= x\sin(\log x) - x\cos(\log x)$

$+ \int x \cdot \frac{-1}{x}\sin(\log x)\,dx$

$= x\sin(\log x) - x\cos(\log x)$

$- \int \sin(\log x)\,dx$

$\therefore 2\int \sin(\log_e x)$

$= x\sin(\log x) - x\cos(\log x)$

$\therefore \int \sin(\ln x) = \frac{x}{2}(\sin$

$= \frac{x}{2}\left(\sin(\log_e x) - \cos(\log_e x)\right)$

Academic Year	Yr 12	Calendar Year	2010
Course	Ext. 2.	Name of task/exam	Trial Exam

Question 2 :

a) $z = \sqrt{3} + i \qquad \omega = 1 - i$

i) $\dfrac{z}{\omega} = \dfrac{\sqrt{3}+i}{1-i}$

$= \dfrac{\sqrt{3}+i}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{\sqrt{3} + \sqrt{3}i + i - 1}{1+1}$

$= \dfrac{(\sqrt{3}-1) + i(\sqrt{3}+1)}{2}$

ii) $|\sqrt{3}+i| = \sqrt{(\sqrt{3})^2 + (1)^2}$

$|z| = 2$

$\arg z = \dfrac{\pi}{6}$

$\therefore z = 2 \operatorname{cis} \dfrac{\pi}{6}$

$|1-i| = \sqrt{(1)^2 + (-1)^2}$

$|w| = \sqrt{2}$

$\arg w = -\dfrac{\pi}{4}$

$\therefore \omega = \sqrt{2} \operatorname{cis}\left(-\dfrac{\pi}{4}\right)$

$\therefore \dfrac{z}{\omega} = \dfrac{2 \operatorname{cis} \frac{\pi}{6}}{\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)}$

$= \sqrt{2} \operatorname{cis}\left(\dfrac{\pi}{6} - -\dfrac{\pi}{4}\right)$

$= \sqrt{2} \operatorname{cis} \dfrac{5\pi}{12}$

iii) $\therefore \sqrt{2} \operatorname{cis} \dfrac{5\pi}{12} = \dfrac{\sqrt{3}-1}{2} + \dfrac{i(\sqrt{3}+1)}{2}$

$\therefore$ equating real parts

$\sqrt{2} \cos \dfrac{5\pi}{12} = \dfrac{\sqrt{3}-1}{2}$

$\therefore \cos \dfrac{5\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

$= \dfrac{\sqrt{3}-1}{2\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$\therefore \cos \dfrac{5\pi}{12} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

iv) $\left(\dfrac{z}{\omega}\right)^{12} = \left(\dfrac{2 \operatorname{cis} \frac{\pi}{6}}{\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)}\right)^{12}$

$= \left(\sqrt{2} \operatorname{cis} \dfrac{5\pi}{12}\right)^{12}$

$= 2^6 \operatorname{cis} 5\pi$

$= 64(\cos 5\pi + i \sin 5\pi)$

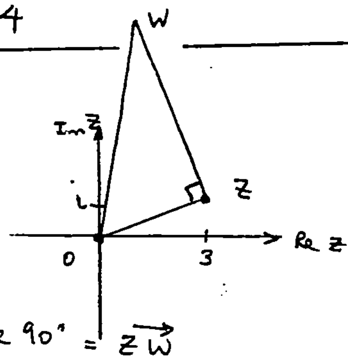
$= 64(-1 + 0i)$

$= -64$

b) $z = 3 + i$

$0 = 0 + 0i$

$\omega = ?$



$2\overrightarrow{zO}$ rotated clockwise $90^\circ = \overrightarrow{zW}$

$2(0 - z)(-i) = \omega - z$

$2(-z)(-i) + z = \omega$

$2zi + z = \omega$

$2(3+i)i + 3 + i = \omega$

$6i - 2 + 3 + i = \omega$

$\omega = 1 + 7i$

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c) $P \;:\; z = x + iy$

$(x+iy)^2 = 2 - (x-iy)^2$

$x^2 + 2xyi - y^2 = 2 - (x^2 - 2xiy - y^2)$

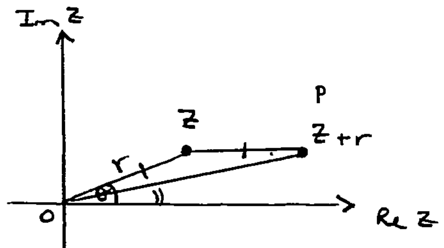
$x^2 + \cancel{2xyi} - y^2 = 2 - x^2 + \cancel{2xyi} + y^2$

$2x^2 - 2y^2 = 2$

$x^2 - y^2 = 1$

$\therefore$ rect. hyperbola.

d)



$\arg z = \theta$.

$\angle OZP = 180 - \theta$ (co-interior angles supplementary on parallel lines).

$\therefore \angle ZOP = \angle ZPO$ (angles opposite equal sides)

$\therefore \angle ZOP = \dfrac{180 - (180 - \theta)}{2}$ (angle sum of triangle)

$= \dfrac{180 - 180 + \theta}{2}$

$= \dfrac{\theta}{2}$

$\therefore \arg(z+r) = \frac{1}{2}\theta$

OR $\arg(z+r) = \arg(r\,\text{cis}\,\theta + r)$

$= \arg(r\cos\theta + r + ir\sin\theta)$

$= \tan^{-1}\left[\dfrac{r\sin\theta}{r(\cos\theta + 1)}\right]$

$= \tan^{-1}\dfrac{\sin\theta}{\cos\theta + 1}$

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

and

$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

So...

$\arg(z+r) = \tan^{-1}\left(\dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} - 1 + 1}\right)$

$= \tan^{-1}\left(\dfrac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}\right)$

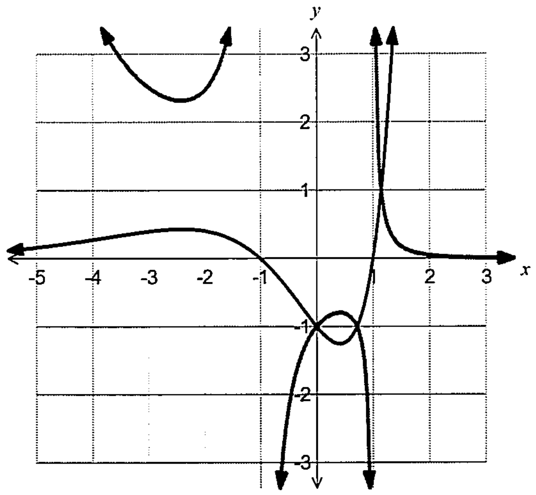
$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$= \frac{\theta}{2}$

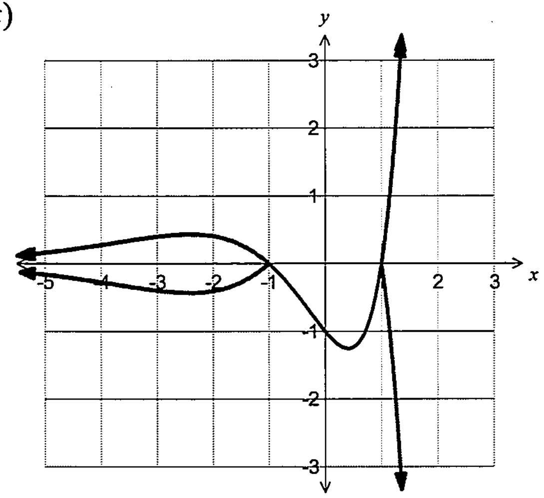
$\therefore \arg(z+r) = \frac{\theta}{2}$.

Question 3 (15 marks) **Start a new sheet of writing paper.** **Marks**

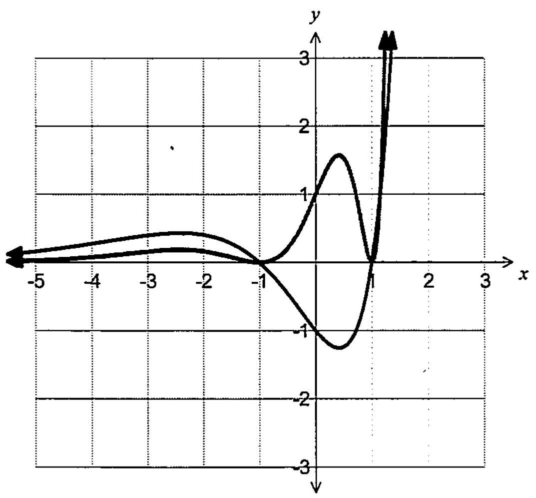
i) $y = \dfrac{1}{f(x)}$ **1**



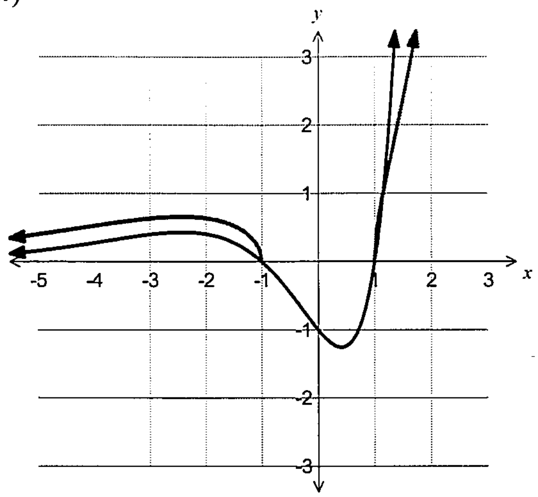
ii) $|y| = f(x)$ **1**



iii) $y = [f(x)]^2$ **2**



iv) $y=\sqrt{f(x)}$



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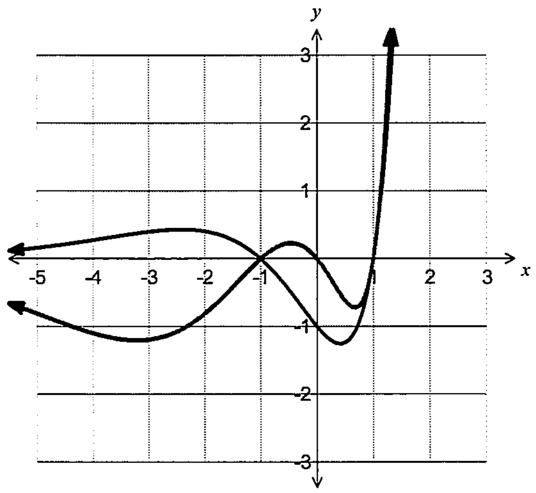
v) $y=x\big(f(x)\big)$



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3 b) i) $1+i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$

ii) $(1+i)^n = 2^{\frac{n}{2}}\operatorname{cis}\frac{\pi n}{4}$

$(1-i)^n = 2^{\frac{n}{2}}\operatorname{cis}\left(-\frac{\pi n}{4}\right)$

$\therefore (1+i)^n + (1-i)^n = 2^{\frac{n}{2}}\operatorname{cis}\frac{\pi n}{4} + 2^{\frac{n}{2}}\operatorname{cis}\left(\frac{-\pi n}{4}\right)$

$= 2^{\frac{n}{2}}\left[\cos\frac{\pi n}{4} + \cancel{i\sin\frac{\pi n}{4}} + \cos\frac{\pi n}{4} - \cancel{i\sin\frac{\pi n}{4}}\right]$

$= 2^{\frac{n}{2}}\left[2\cos\frac{\pi n}{4}\right]$

$= 2\left[2^{\frac{n}{2}}\cos\frac{\pi n}{4}\right]$

iii) $(1+x)^n = p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n$

$(1-x)^n = p_0 - p_1 x + p_2 x^2 - \dots + p_n x^n$

$(1+x)^n + (1-x)^n = 2p_0 + 2p_2 x^2 + 2p_4 x^4 + \dots$
$\dots + 2p_n x^n$

Let $x = i$

$(1+i)^n + (1-i)^n = 2p_0 + 2p_2(-1) + 2p_4(-1)^2 + \dots + 2p_n(-1)$

$2\left[2^{n/2}\cos\frac{\pi n}{4}\right] = 2p_0 - 2p_2 + 2p_4 - \dots$

$2^{n/2}\cos\frac{\pi n}{4} = p_0 - p_2 + p_4 - \dots$

Similarly,

$(1+i)^n - (1-i)^n = 2i\, 2^{n/2}\sin\frac{n\pi}{4}$

and

$(1+x)^n - (1-x)^n = 2p_1 x + 2p_3 x^3 + \dots$

let $x = i$

$(1+i)^n - (1-i)^n = 2p_1 i - 2p_3 i + 2p_5 i - \dots$

$2i\, 2^{n/2}\sin\frac{n\pi}{4} = 2ip_1 - 2ip_3 + 2ip_5 - \dots$

$\therefore 2^{n/2}\sin\frac{n\pi}{4} = p_1 - p_3 + p_5 - \dots$

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Question 4 :

a) i) $\dfrac{(x-1)^2}{16} - \dfrac{(y+3)^2}{25} = 1$

Centre $(1, -3)$

$a = 4 \quad b = 5$

$e^2 = 1 + \left(\frac{5}{4}\right)^2$

$e^2 = 1 + \frac{25}{16}$

$e = \sqrt{\frac{41}{16}}$, $e > 0$

$e = \frac{\sqrt{41}}{4}$

foci: $(1 + \sqrt{41}, -3)$
$(1 - \sqrt{41}, -3)$

ii) directrices :

$x = 1 + \frac{16}{\sqrt{41}}$, $x = 1 - \frac{16}{\sqrt{41}}$

Asymptotes :

$$\frac{(x-1)^2}{16} - 1 = \frac{(y+3)^2}{25}$$

$$\frac{25}{16}(x-1)^2\left[1 - \frac{16}{(x-1)^2}\right] = (y+3)^2$$

$$\therefore y + 3 = \pm\frac{5}{4}(x-1)\sqrt{1 - \frac{16}{(x-1)^2}}$$

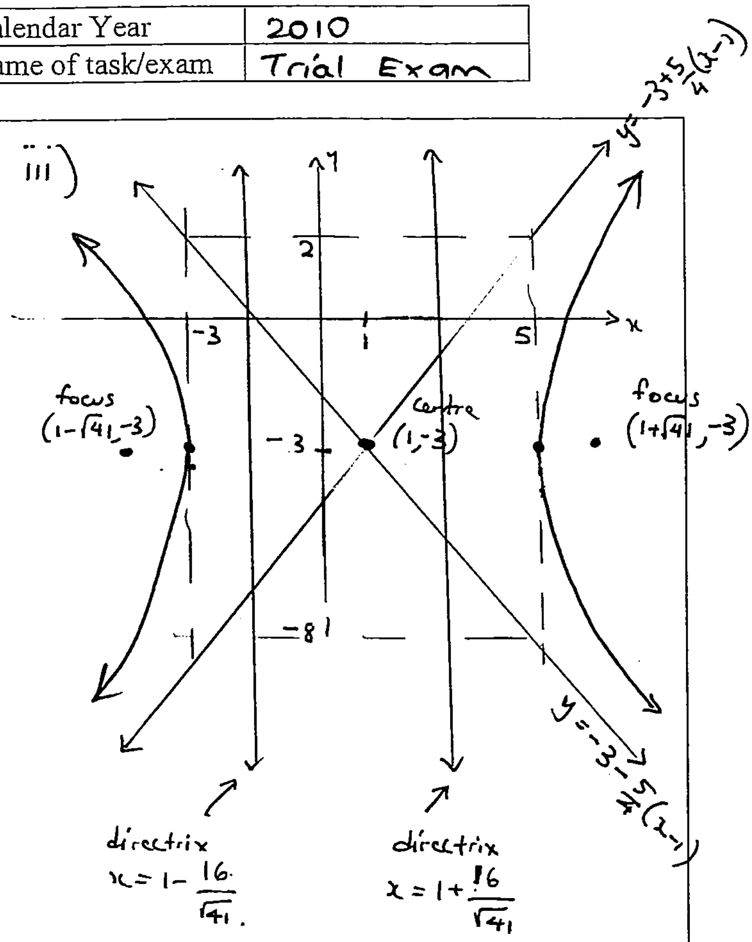
as $x \to \infty$

$$y + 3 = \pm\frac{5}{4}(x-1)$$

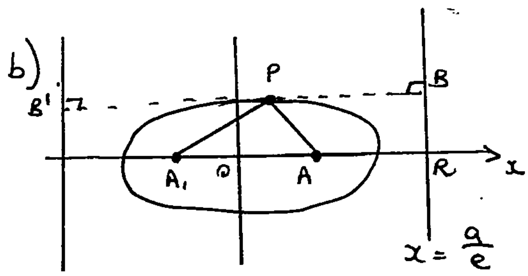
$\therefore$ asymptotes

$$y = -3 \pm \frac{5}{4}(x-1)$$

iii)



b)



$$\frac{PA}{PB} = \frac{PA'}{PB'} = e.$$

$PA = e\,PB \qquad PA' = e\,PB'$

$\therefore PA + PA' = e(PB + PB')$

$= e(2 \times OR)$

$= e\left(2 \times \frac{a}{e}\right)$

$= 2a$

which is a constant.

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c) $x^2 - y^2 = a^2$

i) $(a\sec\theta, a\tan\theta)$.

$$\text{LHS} = (a\sec\theta)^2 - (a\tan\theta)^2$$
$$= a^2(\sec^2\theta - \tan^2\theta)$$
$$= a^2(1)$$
$$= a^2$$
$$= \text{RHS}$$

$\therefore (a\sec\theta, a\tan\theta)$ are the parametric coordinates of a point on the hyperbola $x^2 - y^2 = a^2$

ii) $x^2 - y^2 = a^2$

$$2x - 2y\frac{dy}{dx} = 0$$
$$2x = 2y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{x}{y}$$

at $(a\sec\theta, a\tan\theta)$

$$m_{tang} = \frac{a\sec\theta}{a\tan\theta}$$
$$= \frac{1}{\cos\theta} \div \frac{\sin\theta}{\cos\theta}$$
$$= \frac{1}{\sin\theta}$$

$\therefore$ eqn

$$y - a\tan\theta = \frac{1}{\sin\theta}(x - a\sec\theta)$$
$$y - \frac{a\sin\theta}{\cos\theta} = \frac{1}{\sin\theta}\left(x - \frac{a}{\cos\theta}\right)$$
$$y - \frac{a\sin\theta}{\cos\theta} = \frac{x}{\sin\theta} - \frac{a}{\sin\theta\cos\theta}$$
$$\sin\theta\cos\theta\, y - a\sin^2\theta = x\cos\theta - a$$
$$x\cos\theta - \sin\theta\cos\theta\, y = a - a\sin^2\theta$$
$$= a(1 - \sin^2\theta)$$
$$= a\cos^2\theta$$

$\therefore \div \cos\theta$

$$x - \sin\theta\, y = a\cos\theta$$

iii) asymptotes of $x^2 - y^2 = a^2$ are $y = x$ & $y = -x$.

Consider $y = x$.

$\therefore$ Solve $x - \sin\theta y = a\cos\theta$ & $y = x$

$$x = a\cos\theta + x\sin\theta$$
$$x - x\sin\theta = a\cos\theta$$
$$x(1 - \sin\theta) = a\cos\theta$$

Coord. of Point A.
$$\left\{\begin{array}{l} x = \dfrac{a\cos\theta}{1-\sin\theta} \\ \therefore y = \dfrac{a\cos\theta}{1-\sin\theta} \quad (\text{since } y = x).\end{array}\right.$$

Consider $y = -x$

$$\therefore x = a\cos\theta + \sin\theta(-x)$$

$$x + x\sin\theta = a\cos\theta$$

$$x(1+\sin\theta) = a\cos\theta$$

coords of B

$$x = \frac{a\cos\theta}{1+\sin\theta}$$

$$\therefore y = -\frac{a\cos\theta}{1+\sin\theta} \quad (\text{since } y = -x)$$

$$d_{OA} = \sqrt{\left(\frac{a\cos\theta}{1-\sin\theta} - 0\right)^2 + \left(\frac{a\cos\theta}{1-\sin\theta}\right)^2}$$

$$= \sqrt{2\left(\frac{a\cos\theta}{1-\sin\theta}\right)^2}$$

$$= \sqrt{2}\,\frac{(a\cos\theta)}{1-\sin\theta}$$

similarly

$$d_{OB} = \sqrt{2}\,\frac{(a\cos\theta)}{1+\sin\theta}$$

$$\therefore A = \frac{1}{2}bh$$

$$= \frac{1}{2}\sqrt{2}\,\frac{a\cos\theta}{1-\sin\theta}\cdot\sqrt{2}\,\frac{a\cos\theta}{1+\sin\theta}$$

$$= \frac{a^2\cos^2\theta}{1-\sin^2\theta}$$

$$= a^2\frac{\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}}$$

$$= a^2$$

which is a constant

Solutions for exams and assessment tasks

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Question 5

a) $P(x) = x^4 - 6x^2 - 8x - 3$

$P(-1) = (-1)^4 - 6 - 8(-1) - 3$

$= 1 - 6 + 8 - 3$

$= 0$

$P'(x) = 4x^3 - 12x - 8$

$P'(-1) = -4 + 12 - 8$

$= 0$

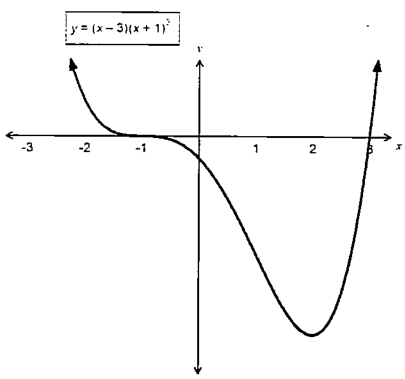
$P''(x) = 12x^2 - 12$

$P''(-1) = 12 - 12$

$= 0$

$\therefore P(x) = (x+1)^3 . (x - a)$

$= (x+1)^3 (x-3)$



b) $P(x) = ax^3 + bx^2 + cx + d$

$P(x) = k(x-5)(x+i)(x-i)$

$= k(x-5)(x^2+1)$

$= k(x^3 + x - 5x^2 - 5)$

$P(x) = k(x^3 - 5x^2 + x - 5)$

$P(2) = 15$

$P(2) = k(8 - 20 + 2 - 5)$

$15 = k(-15)$

$k = -1$

$\therefore P(x) = -x^3 + 5x^2 - x + 5$

$\therefore a = -1$

$b = 5$

$c = -1$

$d = 5$

c) (i) $(1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$

$= (1-\sqrt{x})^{n-1}(1 - (1-\sqrt{x}))$

$= (1-\sqrt{x})^{n-1}(1 - 1 + \sqrt{x})$

$= (1-\sqrt{x})^{n-1}\sqrt{x}$

(ii) $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$

$u = (1-\sqrt{x})^n \qquad v = x$

$du = n(1-\sqrt{x})^{n-1} \cdot -\frac{1}{2}x^{-1/2} \qquad dv = 1$

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$$= \left[x(1-\sqrt{x})^n\right]_0^1 + \frac{n}{2}\int_0^1 \frac{(1-\sqrt{x})^{n-1}}{\sqrt{x}} \cdot x \, dx$$

$$= [0] + \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1} \cdot \sqrt{x} \, dx$$

$$= \frac{n}{2}\int_0^1 (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n \, dx$$

$$= \frac{n}{2} I_{n-1} - \frac{n}{2} I_n$$

$$2I_n = n I_{n-1} - n I_n$$

$$(n+2) I_n = n I_{n-1}$$

$$I_n = \frac{n}{n+2} I_{n-1}$$

(iii) $I_n = \frac{n}{n+2} I_{n-1}$

$$= \frac{n}{n+2} \cdot \frac{n-1}{n+1} I_{n-2}$$

$$= \frac{n}{n+2} \cdot \frac{n-1}{n+1} \cdot \frac{n-2}{n} I_{n-3}$$

$$= \frac{n}{n+2} \cdot \frac{n-1}{n+1} \cdot \frac{n-2}{n} \cdots \frac{2}{4} I_1$$

$$= \frac{n}{n+2} \cdot \frac{n-1}{n+1} \cdot \frac{n-2}{n} \times \cdots \frac{2}{4} \times \frac{1}{3} I_0$$

$$= \frac{n! \; 2!}{(n+2)!}$$

$$\frac{1}{I_n} = \frac{(n+2)!}{n! \; 2!}$$

$$= {}^{n+2}C_2$$

d) $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$

$$\geq 2\sqrt{\frac{a}{b} \cdot \frac{b}{c}} + 2\sqrt{\frac{c}{d} \cdot \frac{d}{a}}$$

$$= 2\sqrt{\frac{a}{c}} + 2\sqrt{\frac{c}{a}}$$

$$= 2\left(\sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}\right)$$

$$= 2\left(2\sqrt{\frac{a}{c} \cdot \frac{c}{a}}\right)$$

$$= 4(1)$$

$$\geq 4$$

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Question 6

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$= \cos^5\theta + 5\cos^4\theta\, i\sin\theta - 10\cos^3\theta\sin^2\theta$

$- 10\cos^2\theta\, i\sin^3\theta + 5\cos\theta\sin^4\theta$

$+ i\sin^5\theta$

Equating imaginary:

$\sin 5\theta = 5\cos^4\theta\sin\theta -$
$10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1-\sin^2\theta)^2.\sin\theta -$
$10(1-\sin^2\theta).\sin^3\theta + \sin^5\theta$

$= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta$
$-10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta$
$-10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

(ii) Let $x = \sin\theta$

$16x^5 - 20x^3 + 5x = 0$

$x(16x^4 - 20x^2 + 5) = 0$

$\sin 5\theta = 0$

$5\theta = 0, \pi, 2\pi, 3\pi, 4\pi$

$\theta = 0\ \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

sol^n to $16x^4 - 20x^2 + 5 = 0$

$x = \sin\frac{\pi}{5}, \sin\frac{2\pi}{5}, \sin\frac{3\pi}{5}$

$\sin\frac{4\pi}{5}$

$\therefore\ x = \sin\frac{k\pi}{5}, k = 1,2,3,4.$

(iii) $y = x+1$

$\therefore x = y-1$

$16(y-1)^4 - 20(y-1)^2 + 5 = 0$

$16(y^4 - 4y^3 + 6y^2 - 4y + 1)$
$-20(y^2 - 2y + 1) + 5 = 0$

$16y^4 - 64y^3 + 96y^2 - 64y + 16$
$-20y^2 + 40y - 20 + 5 = 0$

$16y^4 - 64y^3 + 76y^2 - 24y + 1 = 0$

$\therefore\ 16x^4 - 64x^3 + 76x^2 - 24x + 1 = 0$

(iv) For (iii)

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

$= \frac{\text{roots 3 at a time}}{\text{roots 4 at a time}}$

$$= \frac{24}{16} \div \frac{1}{16}$$

$$= \frac{24}{16} \times \frac{16}{1}$$

$$= 24$$

b) $5x^2 - 6xy + y^2 - 2x + 4y - 3 = 0$

$$10x - 6x\frac{dy}{dx} + y(-6) + 2y\frac{dy}{dx} - 2 + 4\frac{dy}{dx} = 0$$

$$(4 - 6x + 2y)\frac{dy}{dx} = 6y + 2 - 10x$$

$$\frac{dy}{dx} = \frac{6y + 2 - 10x}{4 - 6x + 2y}$$

at $(1, 2)$

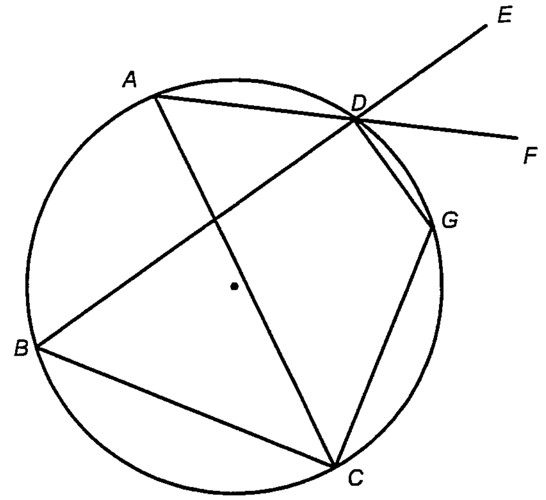
$$\frac{dy}{dx} = \frac{6(2) + 2 - 10}{4 - 6 + 2(2)}$$

$$= \frac{4}{2}$$

$$= 2$$

$$y - 2 = 2(x - 1)$$
$$y - 2 = 2x - 2$$
$$y = 2x$$

c)



let $\angle ACG = x$

$\therefore \angle BCA = x$ (AC bisects $\angle BCG$)

$\therefore \angle ADB = x$ (angles standing on same arc are equal).

$\therefore \angle EDF = x$ (vertically opposite angles equal)

$\angle BDG = 180 - \angle BCG$ (opposite angles in cyclic quad are supplementary)

$= 180 - 2x$

$\therefore \angle FDG = 180 - \angle BDG - \angle ADB$

$= 180 - (180 - 2x) - x$ (angle sum of straight line)

$= 2x - x$

$= x$

$\therefore \angle EDF = \angle FDG$

$\therefore$ FD bisects $\angle EDG$

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Question 7

a) $\frac{d}{dx}(x^n) = n x^{n-1}$

$\frac{d^2}{dx^2}(x^n) = \frac{d}{dx}(n x^{n-1})$

$= n(n-1) x^{n-2}$

$\frac{d^3}{dx^3}(x^n) = n(n-1)(n-2) x^{n-3}$

$\therefore \frac{d^n}{dx^n} x^n = n(n-1)(n-2)\cdots(n-(n+1))x^0$

$= n(n-1)(n-2)\ldots(1)x^0$

$= n!$

(ii) $\frac{d^n}{dx^n}(x^n \ln x) = n!\left(\ln x + \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}\right)$

For $n=1$

$\frac{d}{dx}(x \ln x) = x \cdot \frac{1}{x} + 1 \cdot \ln x$

$= 1 + \ln x$

$= 1!(1 + \ln x)$

Assume true for $n=k$

i.e.

$\frac{d^k}{dx^k}(x^k \ln x)$

$= k!\left(\ln x + \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}\right)$

$\frac{d^{k+1}}{dx^{k+1}}(x^{k+1} \cdot \ln x)$

$= \frac{d}{dx}\left(\frac{d^k}{dx^k}(x^{k+1} \ln x)\right)$

$= \frac{d^k}{dx^k}\left(\frac{d}{dx}(x^{k+1} \cdot \ln x)\right)$

$= \frac{d^k}{dx^k}\left((k+1) x^k \cdot \ln x + x^{k+1} \cdot \frac{1}{x}\right)$

$= \frac{d^k}{dx^k}\left[(k+1) x^k \ln x + x^k\right)$

$= (k+1)\left[\frac{d^k}{dx^k}(x^k \cdot \ln x) + \frac{d^k}{dx^k}(x^k)\right)$

$= (k+1) k!\left(\ln x + \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{k}\right) + k!$

$= (k+1) k!\left(\ln x + \frac{1}{1} + \frac{1}{2} + \ldots \frac{1}{k} + \frac{1}{k+1}\right)$

$= (k+1)!\left[\ln x + \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{k} + \frac{1}{k+1}\right)$

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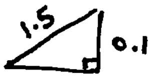
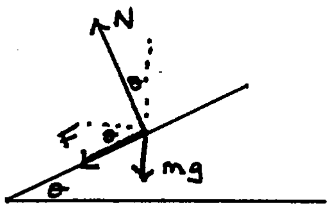
7 b) $r = 500$ m



i)



ii) <u>vert</u>: $N\cos\theta - F\sin\theta - mg = 0$

<u>horiz</u>: $N\sin\theta + F\cos\theta = \frac{mv^2}{r}$

if no sideways force $F = 0$

$\therefore N\cos\theta = mg$

$N\sin\theta = \frac{mv^2}{r}$

$\therefore \tan\theta = \frac{mv^2}{r} \div mg$

$= \frac{\cancel{m}v^2}{r} \times \frac{1}{\cancel{m}g}$

$\therefore \tan\theta = \frac{v^2}{rg}$

iii) we know

by pyth

$x = 1.496\ldots$

$\therefore \tan\theta = \frac{0.1}{1.496\ldots}$

$\therefore v^2 = rg\left(\frac{0.1}{1.496\ldots}\right)$

$v = 18.1$ m/s

c) $+\uparrow$ $\downarrow mg$ $\downarrow R$

i) $m\ddot{x} = -mg - kmv$

$\ddot{x} = -g - kv$

$\ddot{x} = -(g + kv)$

ii) $\frac{dv}{dt} = -(g + kv)$

$\frac{dt}{dv} = -\frac{1}{g + kv}$

$t = \int -\frac{1}{g+kv}\, dv$

$t = -\int \frac{1}{g+kv}\, dv$

$t = -\left[\frac{1}{k}\int \frac{k}{g+kv}\, dv\right]$

$t = -\frac{1}{k}\ln(g + kv) + c$

when $t = 0$ $v = u$.

$\therefore 0 = -\frac{1}{k}\ln(g + ku) + c$

$c = \frac{1}{k}\ln(g + ku)$

$\therefore t = \frac{1}{k}\ln(g + ku) - \frac{1}{k}\ln(g + kv)$

at max height $v = 0$, $t = T$

$\therefore T = \frac{1}{k}\ln(g + ku) - \frac{1}{k}\ln(g)$

$\therefore T = \frac{1}{k}\ln\left(\frac{g + ku}{g}\right)$

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iii) $\ddot{x} = -(g+kv)$

$$v\frac{dv}{dx} = -(g+kv)$$

$$\frac{dv}{dx} = -\frac{(g+kv)}{v}$$

$$\frac{dx}{dv} = -\frac{v}{g+kv}$$

$$x = -\int \frac{v}{g+kv}\,dv$$

$$\begin{array}{r} \frac{1}{k} \\ kv+g\,\big)\,v \\ v+\frac{g}{k} \\ \hline -\frac{g}{k} \end{array}$$

$$x = -\int\left(\frac{1}{k} - \frac{\frac{g}{k}}{kv+g}\right)dv$$

$$x = -\int\left(\frac{1}{k} - \frac{g}{k}\,\frac{1}{kv+g}\right)dv$$

$$x = -\int\left(\frac{1}{k} - \frac{g}{k^2}\,\frac{k}{kv+g}\right)dv$$

$$x = -\left[\frac{1}{k}v - \frac{g}{k^2}\ln(kv+g)\right]+c$$

when $v=u$ $x=0$

$$0 = -\left[\frac{u}{k} - \frac{g}{k^2}\ln(ku+g)\right]+c$$

$$c = \frac{u}{k} - \frac{g}{k^2}\ln(g+ku)$$

$$\therefore x = -\frac{1}{k}v + \frac{g}{k^2}\ln(kv+g) + \frac{u}{k} - \frac{g}{k^2}\ln(g+ku)$$

at max $v=0$, $x=h$

$$\therefore h = 0 + \frac{g}{k^2}\ln g + \frac{u}{k} - \frac{g}{k^2}\ln(g+ku)$$

$$h = \frac{1}{k}\left[u + \frac{g}{k}\ln\left(\frac{g}{g+ku}\right)\right]$$

$$= \frac{1}{k}\left[u + g(-T)\right]$$

since $Tk = \ln\left(\frac{g+ku}{g}\right)$

$$-Tk = -\ln\left(\frac{g+ku}{g}\right)$$

$$-Tk = \ln\left(\frac{g}{g+ku}\right)$$

$$-T = \frac{1}{k}\ln\left(\frac{g}{g+ku}\right)$$

$$\therefore h = \frac{1}{k}\left[u-gT\right]$$

Question 8:

a) i) $\ddot{y} = -g$

$$\dot{y} = -g\int dt$$

$$\dot{y} = -gt + c$$

$$u\sin\alpha = 0 + c_1$$

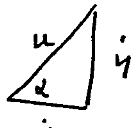
$$\therefore \dot{y} = -gt + u\sin\alpha$$

$$y = -\frac{1}{2}gt^2 + ut\sin\alpha + c_2$$

$$0 = 0 + 0 + c_2$$

$$\therefore y = -\frac{1}{2}gt^2 + ut\sin\alpha$$

Initial Conditions



$\dot{x} = u\cos\alpha$

$\dot{y} = V\sin\alpha$

$t=0$

$x=0$

$y=0$

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$\ddot{x} = 0$

$\dot{x} = c_3$

$u\cos\alpha = c_3$

$\therefore \dot{x} = u\cos\alpha$

$x = ut\cos\alpha + c_4$

$0 = 0 + c_4$

$\therefore x = ut\cos\alpha$

Cartesian equation:

$t = \dfrac{x}{u\cos\alpha}$

$y = -\frac{1}{2}g\left(\dfrac{x}{u\cos\alpha}\right)^2 + u\left(\dfrac{x}{u\cos\alpha}\right)\sin\alpha$

$y = -\frac{1}{2}g\dfrac{x^2}{u^2\cos^2\alpha} + x\tan\alpha$

$y = \dfrac{-gx^2}{2u^2}\sec^2\alpha + x\tan\alpha$

ii) (a, b) satisfies

$b = \dfrac{-ga^2}{2u^2}(\tan^2\alpha + 1) + a\tan\alpha$

$2u^2b = -ga^2\tan^2\alpha - ga^2 + 2u^2a\tan\alpha$

$ga^2\tan^2\alpha - 2u^2a\tan\alpha + (2u^2b + ga^2) = 0$

if 2 trajectoris $\Delta > 0$

$b^2 - 4ac > 0$

$(-2u^2a)^2 - 4(ga^2)(2u^2b + ga^2) > 0$

$4u^4a^2 - 8ga^2bu^2 - 4g^2a^4 > 0$

$\div 4a^2$

$u^4 - 2gbu^2 - g^2a^2 > 0$

$\left[(u^2 - gb)^2 = u^4 - 2u^2gb + g^2b^2\right]$

$\therefore (u^2 - gb)^2 - g^2a^2 - g^2b^2 > 0$

$(u^2 - gb)^2 > g^2a^2 + g^2b^2$

$(u^2 - gb)^2 > g^2(a^2 + b^2)$

b) i) Vert: $T\cos\alpha + N\sin\alpha - mg = 0$

$T\cos\alpha + N\sin\alpha = mg$ (1)

horiz: $T\sin\alpha - N\cos\alpha = mr\omega^2$

$\therefore \cos\frac{\pi}{3} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore T\left(\frac{1}{2}\right) + N\left(\frac{\sqrt{3}}{2}\right) = mg$

$T + \sqrt{3}N = 2mg$ (1)

$T\left(\frac{\sqrt{3}}{2}\right) - N\left(\frac{1}{2}\right) = mr\omega^2$

$\sqrt{3}T - N = 2mr\omega^2$ (2)

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ii) $N = \sqrt{3}T - 2mr\omega^2$

$\therefore T + \sqrt{3}\left[\sqrt{3}T - 2mr\omega^2\right] = 2mg$

$T + 3T - 2\sqrt{3}\,mr\omega^2 = 2mg$

$4T = 2mg + 2\sqrt{3}\,mr\omega^2$

$T = \frac{m}{4}\left[2g + 2\sqrt{3}\,r\omega^2\right]$

we know

$\therefore \sin 60 = \frac{r}{5}$

$r = \frac{5\sqrt{3}}{2}$.

$\therefore T = \frac{m}{4}\left[2g + 2\sqrt{3}\cdot\frac{5\sqrt{3}}{2}\omega^2\right]$

$\therefore T = \frac{m}{4}\left[2g + 15\omega^2\right]$

$N = \sqrt{3}T - 2mr\omega^2$

$= \sqrt{3}\left[\frac{m}{4}\left(2g + 15\omega^2\right)\right] - 2m\frac{5\sqrt{3}}{2}\omega^2$

$= \sqrt{3}\left[\frac{2mg}{4} + \frac{15m\omega^2}{4}\right] - 5\sqrt{3}m\omega^2$

$= \frac{\sqrt{3}}{4}\left[2mg + 15m\omega^2 - 20m\omega^2\right]$

$\therefore N = \frac{\sqrt{3}}{4}m\left(2g - 5\omega^2\right)$

iii) need $N > 0$

$\frac{2\sqrt{3}mg}{4} - \frac{5\sqrt{3}}{4}m\omega^2 > 0$

$2\sqrt{3}g - 5\sqrt{3}\omega^2 > 0$

$5\sqrt{3}\omega^2 - 2\sqrt{3}g < 0$

$\therefore \omega^2 < \frac{2\sqrt{3}g}{5\sqrt{3}}$

$\therefore \omega^2 < \frac{2g}{5}$

end of exam